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A ray method for near-shore plumes in a shallow-water flow

By RONALD SMITH
Department of Mathematical Sciences, University of Technology, Loughborough, LE11 3TU, UK

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Far from a shoreline, the spreading of a contaminant plume in a shallow-water flow can be predicted easily and accurately by a ray-tracing method. Unfortunately, the concentration predictions become singular at a beach, where the ray paths have a caustic. In this paper a uniform approximation is derived which remains valid at a beach. It is shown how the singular ray solutions corresponding to rays incident to and transmitted from the beach can be combined to construct the uniform approximation.

1. Introduction

Ray methods are usually associated with the propagation of short waves in non-uniform media. The rays are the principal direction of information (or energy) propagation, and along these ray paths it is an elementary computational task to evaluate the phase of the waves. Also, the square of the wave amplitude varies inversely with the separation between rays. Cohen & Lewis (1967) pointed out that the essential ingredient is that the solution varies much more rapidly than the non-uniform medium, and they showed how ray methods could be used to solve diffusion equations. Instead of rapid phase oscillations there is rapid Gausian (exponential) decay, but otherwise the calculations proceed the same as for wave equations.

Contaminant plumes in shallow-water flows are very long and narrow. A photograph to illustrate the narrowness of plumes is given by Fischer *et al.* (1979, figure 5.5). So, across the plume there is comparatively rapid decay of the contaminant concentration. The application of ray methods to shallow-water contaminant plumes has been investigated by Smith (1981) and by Kay (1987). The relationship between the ray, flow and flux directions is that the flux direction is midway between the ray and flow directions (see figure 1 of Smith 1981).

At a beach the flux, flow and rays become parallel to the shoreline. Thus, the separation between adjacent ray paths tends to zero at the beach (see figure 1) and the amplitude of the ray solution becomes erroneously singular. Smith (1983) has given a modified ray calculation which only gives the concentration at the shoreline. The purpose of the present paper is to derive a uniform approximation which is valid both far from and close to the shoreline.

For wave problems there are well-established methods for the construction of non-singular solutions when the ray paths have caustics. A local analysis near the caustic can be used to derive an inner solution which can then be matched to an outer ray solution (Buchal & Keller 1960; Keller & Rubinow 1960). Alternatively, the inner solution can be stretched and twisted so that it remains accurate at all distances from

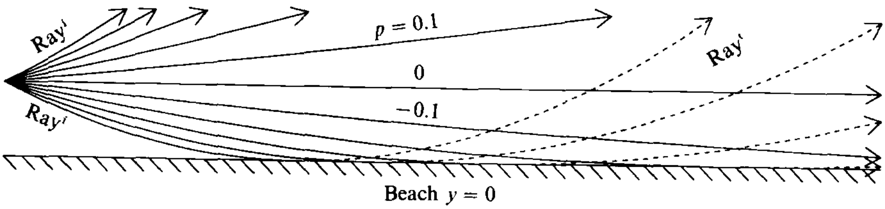


FIGURE 1. A sketch of ray paths incident to and transmitted from a beach.

the caustic (Ludwig 1966; Smith 1970). A convenient feature of the uniform solutions is that all the necessary information can be retrieved from the singular ray solutions for the two categories of rays approaching or leaving the caustic.

For the diffusion problem of a contaminant plume adjacent to a uniformly sloping beach, Kay (1987) has derived a solution which can be interpreted as being the inner solution. Here it is shown how the uniform construction can be adapted to extend Kay's (1987) solution that it is accurate at all distances from the shoreline. The prinicpal restriction is that near the shoreline the beach slope is neither zero nor infinite. As in the wave case, all the necessary information can be retrieved from the singular ray solutions for the two categories of rays approaching or leaving the beach.

2. Ray solution

For a shallow-water flow the (symmetric) horizontal diffusivity tensor $\epsilon \boldsymbol{D}$ scales as the product of the local water depth h and the local friction velocity u_* (Elder 1959). The horizontal lengthscale L for changes in the flow or bed topography is typically very large relative to a refernce depth H. Similarly, a reference bulk velocity U is large relative to a reference friction velocity U_*. The smallness of the horizontal diffusivity tensor $\epsilon \boldsymbol{D}$ relative to Lu can be accounted for by our interpreting ϵ as being a small parameter.

The vertically integrated advection–diffusion equation takes the form

$$\pm h\boldsymbol{u}\cdot\nabla c-\epsilon\nabla\cdot(h\boldsymbol{D}\cdot\nabla c)=0, \tag{2.1a}$$

with

$$\nabla\cdot(h\boldsymbol{u})=0. \tag{2.1b}$$

Here $\boldsymbol{u}$ is the horizontal current, ∇ is the horizontal gradient opertor and c is the vertically averaged concentration. The $\pm$ sign is a technical device which is used to ensure that the solution for the concentration has the correct asymmetry if the flow were to be reversed (Smith 1981).

The ray solution takes the form

$$c=\epsilon^{-\frac{1}{2}}z\exp(\pm\epsilon^{-1}\phi) \tag{2.2}$$

(Smith 1981, equation (2.2)). The rapid Gaussian (exponential) decay across the contaminant plume is accommodated via the explicit ϵ^{-1} factor in the exponential. So, although the concentration c varies rapidly, the amplitude factor z and the decay exponent ϕ are assumed to vary on the lengthscale L (except near the source). For wave problems the $\pm$ sign would be replaced by i.

If we substitute the ansatz (2.2) into the reversible advection–diffusion equation (2.1a), then we generate two classes of terms proportionsl to

$$\exp(\pm\epsilon^{-1}\phi), \quad \pm\exp(\pm\epsilon^{-1}\phi). \tag{2.3}$$

Equating these classes of terms separately to zero, we have

$$h\boldsymbol{u}\cdot\nabla\phi - h\nabla\phi\cdot\boldsymbol{D}\cdot\nabla\phi = \epsilon^2\nabla\cdot(h\boldsymbol{D}\cdot\nabla z)/z, \tag{2.4a}$$

$$h\boldsymbol{u}\cdot\nabla z - \nabla\cdot(zh\boldsymbol{D}\cdot\nabla\phi) - h\nabla\phi\cdot\boldsymbol{D}\cdot\nabla z = 0. \tag{2.4b}$$

Again, we remark that for wave problems the same splitting would be achieved by taking real and imaginary parts.

Away from the source, the amplitude z and the decay exponent ϕ are assumed to vary on the horizontal lengthscale L for changes in the flow. So, for small diffusion $\epsilon\boldsymbol{D}$ we can neglect the ϵ^2 term on the right-hand side of (2.4*a*). The resulting first-order equation for ϕ is

$$\boldsymbol{u}\cdot\nabla\phi - \nabla\phi\cdot\boldsymbol{D}\cdot\nabla\phi = 0. \tag{2.5}$$

In wave terminology, this would be called the dispersion relation. There are special directions (bi-characteristics or rays)

$$\boldsymbol{t} = (\boldsymbol{u} - 2\boldsymbol{D}\cdot\nabla\phi)/|\boldsymbol{u}|, \tag{2.6}$$

along which the partial differential equation (2.5) becomes a first-order ordinary differential equation (Smith 1981, §4). Also, if J denotes the separation between adjacent rays, then the amplitude equation (2.4*b*) can be integrated:

$$z^2 h\,|\boldsymbol{u}|\,J = \text{constant along rays} \tag{2.7}$$

(Smith 1981, §5). Hence, as widely used in the context of water waves, ray tracing gives an easy computational procedure for calculating the concentration away from the beaches. (Section 6 details a simplified version of the ray calculations in which only the component of $\epsilon\boldsymbol{D}$ across the flow is retained.)

Alas, at a beach the water depth h, flow speed $\boldsymbol{u}$ and ray separation J all tend to zero. For turbulent flow along a smooth beach, with non-zero but finite slope, the respective rates of decay with off-shore distance y are $y, y^{\frac{1}{2}}$ and $y^{\frac{1}{2}}$ (see §6). The combined effect is to give the amplitude z a y^{-1} singularity at the beach. The source of this localized error in the ray solution is that there is rapid variation near the beach.

3. Kay's solution for a plume on a uniformly sloping beach

For flow along a straight uniformly sloping beach, and with longitudinal (or skew) diffusion neglected, Kay (1987, equation (35)) derived an exact solution. A representation of Kay's solution which highlights the different lengthscales involved is

$$c = \epsilon^{-\frac{5}{2}} A \exp(\pm\epsilon^{-1}\sigma) K(\epsilon^{-2}\rho), \tag{3.1a}$$

where

$$K(\xi) = \frac{2}{\xi}\left\{\cosh\xi^{\frac{1}{2}} - \frac{\sinh\xi^{\frac{1}{2}}}{\xi^{\frac{1}{2}}}\right\} = \frac{(2\pi)^{\frac{1}{2}}}{\xi^{\frac{3}{4}}} I_{\frac{3}{2}}(\xi^{\frac{1}{2}}). \tag{3.1b}$$

Here A is an algebraic amplitude factor, which remains finite at the beach, and σ, ρ are similarity variables. The beach position corresponds to $\rho = 0$. The function $K(\xi)$ satisfies the second-order ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^{\frac{5}{2}}\frac{\mathrm{d}}{\mathrm{d}\xi}K\right) - \tfrac{1}{4}\xi^{\frac{3}{2}}K = 0, \tag{3.2a}$$

with

$$K(0) = \tfrac{2}{3}. \tag{3.2b}$$

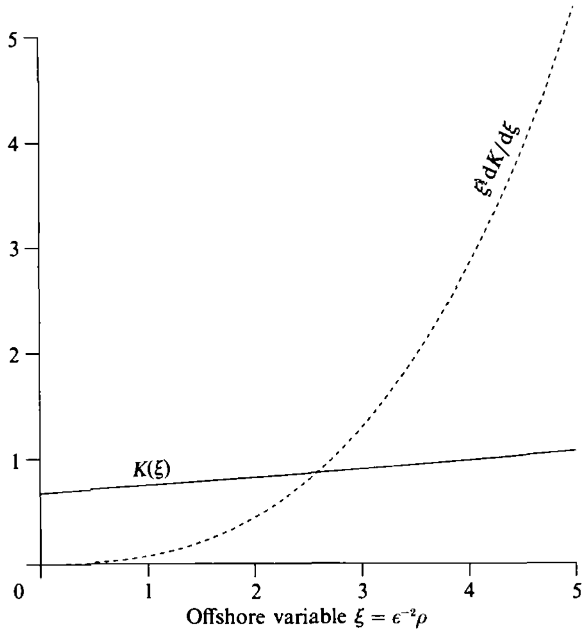


FIGURE 2. Graph of the functions $K(\xi)$ and $\xi^{\frac{5}{2}}\mathrm{d}K/\mathrm{d}\xi$ involved in Kay's (1987) exact solution and in its distorted generalization.

The concentration remains finite at the beach (see figure 2), but away from the beach there is rapid dependence upon the similarity variable ρ.

On a short enough lengthscale (of order $\epsilon^2 L$), any smooth beach with non-zero but finite slope looks straight and uniform. Also, except near the source, the disparity between (large) advection and (small) diffusion implies that longitudinal diffusion is negligible. So, for a general beach, Kay's (1987) solution could be used as an inner solution as in the work of Keller and his collaborators (Buchal & Keller 1960; Keller & Rubinow 1960). Instead, we shall use Kay's (1987) solution as the basic building-block for a uniformly valid solution.

It is possible to envisage beach geometries which have zero or infinite beach slopes at the shoreline, or singularities away from the beach. For such geometries Kay's solution would not be the appropriate inner solution. However, if there were an inner solution of separation-of-variables types, then it would be possible to adapt the subsequent analysis to accommodate the different class of beach geometries.

4. Uniform solutions

Conveniently, the structure ((3.1*a*), (3.2*a*)) of Kay's solution conforms precisely to the requirements ((1.2.1), (1.2.2)) needed for Smith's (1970) method for constructing uniform asymptotic solutions. With $\pm$ replacing i, the counterpart to equation (1.2.3) of Smith (1970) is the representation

$$c = \epsilon^{-\frac{5}{2}} \exp(\pm\theta) \left\{ AK(\xi) \pm \epsilon^4 B \xi^{\frac{5}{2}} \frac{\mathrm{d}K}{\mathrm{d}\xi} \right\}, \tag{4.1a}$$

with

$$\theta = \frac{\sigma}{\epsilon}, \quad \xi = \frac{\rho}{\epsilon^2}. \tag{4.1b}$$

Here the unknown amplitudes A, B and unknown variables σ, ρ are assumed to vary on the horizontal lengthscale L. By construction, the ansatz (4.1*a*, *b*) inherits from Kay's (1987) solution the regularity property at the shoreline.

If we substitute the ansatz (4.1*a*, *b*) into the reversible advection–diffusion equation (2.1*a*), then we generate 4 classes of terms:

$$\exp(\pm\theta)K, \quad \pm\exp(\pm\theta)\,\xi^{\frac{5}{2}}\frac{\mathrm{d}K}{\mathrm{d}\xi}, \quad \pm\exp(\pm\theta)K, \quad \exp(\pm\theta)\,\xi^{\frac{5}{2}}\frac{\mathrm{d}K}{\mathrm{d}\xi}. \tag{4.2}$$

Proceeding as in Smith (1970), we argue that to preserve the asymmetry for reversed flow, and because of the different rapid variations of K and of $\xi^{\frac{5}{2}}\mathrm{d}K/\mathrm{d}\xi$, all four categories of terms must separately be zero:

$$0 = hA\left[\boldsymbol{u}\cdot\nabla\sigma - \nabla\sigma\cdot\boldsymbol{D}\cdot\nabla\sigma - \frac{1}{4\rho}\nabla\rho\cdot\boldsymbol{D}\cdot\nabla\rho\right] + \tfrac{1}{4}\rho^{\frac{3}{2}}hB[\boldsymbol{u}\cdot\nabla\rho - 2\nabla\sigma\cdot\boldsymbol{D}\cdot\nabla\rho] - \epsilon^2\nabla\cdot(h\boldsymbol{D}\nabla A), \tag{4.3a}$$

$$0 = hB\left[\boldsymbol{u}\cdot\nabla\sigma - \nabla\sigma\cdot\boldsymbol{D}\cdot\nabla\sigma - \frac{1}{4\rho}\nabla\rho\cdot\boldsymbol{D}\cdot\nabla\rho\right] + \frac{hA}{\rho^{\frac{5}{2}}}[\boldsymbol{u}\cdot\nabla\rho - 2\nabla\sigma\cdot\boldsymbol{D}\cdot\nabla\rho] - \epsilon^2\nabla\cdot(h\boldsymbol{D}\cdot\nabla B), \tag{4.3b}$$

$$0 = h\boldsymbol{u}\cdot\nabla A - \nabla\cdot(Ah\boldsymbol{D}\cdot\nabla\sigma) - h\nabla\sigma\cdot\boldsymbol{D}\cdot\nabla A - \tfrac{1}{4}\nabla\cdot(\rho^{\frac{3}{2}}Bh\boldsymbol{D}\cdot\nabla\rho) - \tfrac{1}{4}\rho^{\frac{3}{2}}h\nabla\rho\cdot\boldsymbol{D}\cdot\nabla B, \tag{4.3c}$$

$$0 = h\boldsymbol{u}\cdot\nabla B - \nabla\cdot(Bh\boldsymbol{D}\cdot\nabla\sigma) - h\nabla\sigma\cdot\boldsymbol{D}\cdot\nabla B - \nabla\cdot\left(\frac{Ah}{\rho^{\frac{5}{2}}}\boldsymbol{D}\cdot\nabla\rho\right) - \frac{h}{\rho^{\frac{5}{2}}}\nabla\rho\cdot\boldsymbol{D}\cdot\nabla A. \tag{4.3d}$$

It deserves emphasis that these equations (4.3*a*–*d*) for the four unknowns A, B, ρ, σ are exact. No use has yet been made of the smallness of the diffusion $\epsilon\boldsymbol{D}$ relative to longitudinal advection.

Away from the source A, B, ρ σ are assumed to vary on the horizontal lengthscale L for changes in the flow, and not the shorter lengthscales associated with the weak diffusion $\epsilon\boldsymbol{D}$. So, for small ϵ we infer that

$$A \sim A^{(0)} + \epsilon^2 A^{(1)} + \ldots \quad B \sim B^{(0)} + \epsilon^2 B^{(1)} + \ldots, \tag{4.4a, b}$$

$$\rho \sim r + \epsilon^2 r^{(1)} + \ldots, \quad \sigma \sim s + \epsilon^2 s^{(1)} + \ldots, \tag{4.4c, d}$$

where $A^{(j)}$, $B^{(j)}$, $r^{(j)}$, $s^{(j)}$ are all independent of ϵ. The leading-order equations are

$$0 = \boldsymbol{u}\cdot\nabla s - \nabla s\cdot\boldsymbol{D}\cdot\nabla s - \frac{1}{4r}\nabla r\cdot\boldsymbol{D}\cdot\nabla r, \tag{4.5a}$$

$$0 = \boldsymbol{u}\cdot\nabla r - 2\nabla s\cdot\boldsymbol{D}\cdot\nabla r, \tag{4.5b}$$

$$0 = h\boldsymbol{u}\cdot\nabla A^{(0)} - \nabla\cdot(A^{(0)}h\boldsymbol{D}\cdot\nabla s) - h\nabla s\cdot\boldsymbol{D}\cdot\nabla A^{(0)} - \tfrac{1}{4}\nabla\cdot(r^{\frac{3}{2}}B^{(0)}h\boldsymbol{D}\cdot\nabla r) - \tfrac{1}{4}r^{\frac{3}{2}}h\nabla r\cdot\boldsymbol{D}\cdot\nabla B^{(0)}, \tag{4.5c}$$

$$0 = h\boldsymbol{u}\cdot\nabla B^{(0)} - \nabla\cdot(B^{(0)}h\boldsymbol{D}\cdot\nabla s) - h\nabla s\cdot\boldsymbol{D}\cdot\nabla B^{(0)} - \nabla\cdot\left(\frac{A^{(0)}h}{r^{\frac{5}{2}}}\boldsymbol{D}\cdot\nabla r\right) - \frac{h}{r^{\frac{5}{2}}}\nabla r\cdot\boldsymbol{D}\cdot\nabla A^{(0)}. \tag{4.5d}$$

If ϵ is a genuine measure of the narrowness of the contaminant plume, then for the one-term truncation to be accurate it suffices that ϵ^2 be small. Far downstream the plume does eventually become wide. The growing width corresponds to the variables r, s both decreasing. In particular, where both r decreases to order ϵ and s decreases to order ϵ^2, there is no longer the required rapid variation and the representation (4.1a) ceases to be appropriate.

5. Relation to ray methods

Away from the shoreline, where $\rho = 0$, we can use the large-ξ asymptotes

$$K \sim \frac{1}{\xi}\exp(\xi^{\frac{1}{2}}), \quad \xi^{\frac{5}{2}}\frac{\mathrm{d}K}{\mathrm{d}\xi} \sim \xi\exp(\xi^{\frac{1}{2}}), \tag{5.1a, b}$$

to convert the uniform representation (4.1a, b) into a ray approximation

$$c \sim \epsilon^{-\frac{1}{2}} z^{\mathrm{i}} \exp\left(\pm\frac{\phi^{\mathrm{i}}}{\epsilon}\right) + \epsilon^{-\frac{1}{2}} z^{\mathrm{t}} \exp\left(\pm\frac{\phi^{\mathrm{t}}}{\epsilon}\right), \tag{5.2a}$$

with
$$\phi^{\mathrm{i}} = s + r^{\frac{1}{2}}, \quad z^{\mathrm{i}} = \frac{A^{(0)}}{r} + rB^{(0)}, \tag{5.2b, c}$$

$$\phi^{\mathrm{t}} = s - r^{\frac{1}{2}}, \quad z^{\mathrm{t}} = \frac{A^{(0)}}{r} - rB^{(0)}. \tag{5.2d, e}$$

We can associate $\phi^{\mathrm{i}}, z^{\mathrm{i}}$ with rays going from the source prior to reaching the shoreline, and $\phi^{\mathrm{t}}, s^{\mathrm{t}}$ with the transmitted rays returning from the shoreline (see figure 1). As we might expect, the ray combinations (5.2 b–e) satisfy the ray equations

$$\boldsymbol{u}\cdot\nabla\phi^{\mathrm{i}} - \nabla\phi^{\mathrm{i}}\cdot\boldsymbol{D}\cdot\nabla\phi^{\mathrm{i}} = 0, \tag{5.3a}$$

$$h\boldsymbol{u}\cdot\nabla z^{\mathrm{i}} - \nabla\cdot(h\boldsymbol{D}z^{\mathrm{i}}\nabla\phi^{\mathrm{i}}) - h\nabla\phi^{\mathrm{i}}\cdot\boldsymbol{D}\cdot\nabla z^{\mathrm{i}} = 0, \tag{5.3b}$$

with equivalent equations for $\phi^{\mathrm{t}}, z^{\mathrm{t}}$.

In practice it is easier to compute the ray solutions than the uniform solutions (see §6). So, we invert the relationships (5.2b–e) and we use the singular ray solutions to construct the uniform solution:

$$s = \tfrac{1}{2}(\phi^{\mathrm{i}} + \phi^{\mathrm{t}}), \quad r = \frac{1}{4}(\phi^{\mathrm{i}} - \phi^{\mathrm{t}})^2, \tag{5.4a, b}$$

$$A^{(0)} = \frac{r}{2}(z^{\mathrm{i}} + z^{\mathrm{t}}), \quad B^{(0)} = \frac{1}{2r}(z^{\mathrm{i}} - z^{\mathrm{t}}). \tag{5.4c, d}$$

At the shoreline the matching of ϕ and z between the two branches ensures that $r = 0$. Thus, although the ray amplitudes $z^{\mathrm{i}}, z^{\mathrm{t}}$ are singular at the shoreline, the amplitudes $A^{(0)}, B^{(0)}$ for the uniform solution are non-singular.

6. Ray tracing in flow-following coordinates

If we retain only tranverse diffusion (i.e. we neglect longitudinal and skew compents of the horizontal diffusion tensor $\boldsymbol{D}$), then the ray calculations can be performed particularly neatly in a generalized coordinate system x, y aligned along and across the flow (Smith 1983). We use the notation m_1, m_2 to denote the metric coefficients in the relationship

$$\mathrm{d}s^2 = m_1^2\,\mathrm{d}x^2 + m_2^2\,\mathrm{d}y^2 \tag{6.1}$$

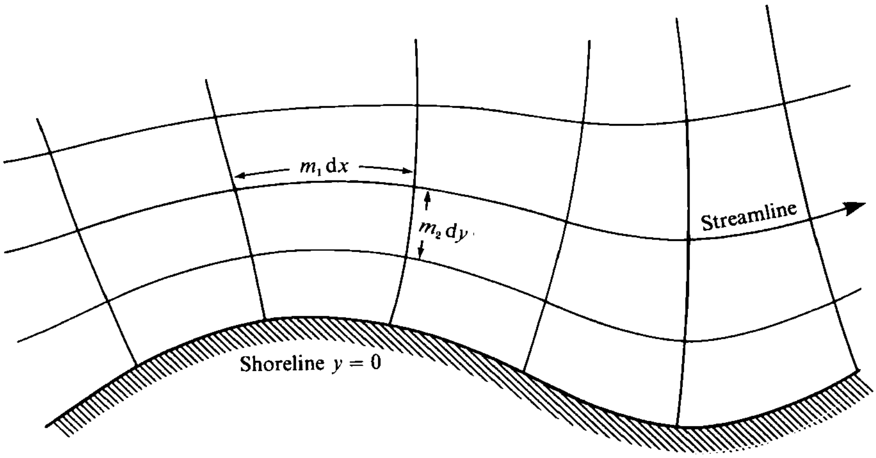


FIGURE 3. Sketch of the along- and across-flow coordinate system.

between the distance increment ds and the coordinate increments dx, dy (see figure 3). Thus, the eikonal equation (2.5) for the decay exponent ϕ becomes

$$\partial_x \phi - F(\partial_y \phi)^2 = 0, \tag{6.2a}$$

with

$$F = \frac{m_1 D_{22}}{m_2^2 u}. \tag{6.2b}$$

For turbulent flow along a smooth beach the diffusivity/velocity (and metric coefficient) ratio F is proportional to the local water depth. The value of this ratio at the discharge position x_0, y_0 is denoted F_0.

The bi-characteristic or ray direction is along the vector

$$\left(m_1, m_2\left(\frac{F}{F_0}\right)^{\frac{1}{2}}\delta\right) \quad \text{with} \quad \delta = -2(FF_0)^{\frac{1}{2}}\partial_y \phi. \tag{6.3a, b}$$

We make a change of variables from (x, y) to (X, p) where p is the initial value of δ and X is the increment of x along an individual ray as it goes outwards from the discharge position x_0, y_0. Thus, differentiation along a ray is defined

$$\partial_X = \partial_x + \left(\frac{F}{F_0}\right)^{\frac{1}{2}} \delta\, \partial_y. \tag{6.4}$$

Along rays the equations for y, δ and ϕ take the neat forms

$$\partial_X y = \left(\frac{F}{F_0}\right)^{\frac{1}{2}} \delta, \quad \partial_X \delta = \frac{\delta}{2F}\partial_x F, \quad \partial_X \phi = -\frac{\delta^2}{4F_0}. \tag{6.5a–c}$$

The initial conditions at the discharge site are

$$y = y_0, \quad \delta = p, \quad \phi = 0 \quad \text{at} \quad X = 0, \tag{6.6a–c}$$

where the parameter p is used to label the individual rays. For later use we note that the separation $J = \partial_p y$ between adjacent rays satisfies the equation

$$\partial_X(J) = \partial_X \partial_p y = \partial_p \partial_X y = (\partial_p x \partial_x + \partial_p y \partial_y)\partial_X y = J\partial_y\left(\left(\frac{F}{F_0}\right)^{\frac{1}{2}}\delta\right). \tag{6.7}$$

To extend rays δ^{i}, ϕ^{i} incident to the shoreline caustic $y=0$, into rays δ^{t}, ϕ^{t} transmitted from the shoreline, it suffices that the orientation parameter δ is reversed and the decay exponent ϕ is continuous:

$$\delta^{t} = -\delta^{i}, \quad \phi^{t} = \phi^{i} \quad \text{at the shoreline} \quad y = 0. \tag{6.8a, b}$$

For the special case in which the diffusivity/velocity (and metric coefficient) ratio F is independent of the longshore coordinate x, it is elementary to solve equations (6.5) and (6.6):

$$\int_{y_0}^{y} \left(\frac{F_0}{F}\right)^{\frac{1}{2}} \mathrm{d}y' = pX, \quad \delta^{i} = p, \quad \phi^{i} = -\frac{p^2}{4F_0} X, \tag{6.9a–c}$$

$$J^{i} = X\left(\frac{F}{F_0}\right)^{\frac{1}{2}}. \tag{6.9d}$$

So, near the shoreline $y=0$, J^{i} tends to zero at the rate $y^{\frac{1}{2}}$. Beyond the caustic the extended solution (for p negative) is

$$\int_{0}^{y} \left(\frac{F_0}{F}\right)^{\frac{1}{2}} \mathrm{d}y' + \int_{0}^{y_0} \left(\frac{F_0}{F}\right)^{\frac{1}{2}} \mathrm{d}y' = -pX, \quad \delta^{t} = -p, \tag{6.10a, b}$$

$$\phi^{t} = -\frac{p^2}{4F_0} X, \quad J^{t} = -X\left(\frac{F}{F_0}\right)^{\frac{1}{2}}. \tag{6.10c, d}$$

If we eliminate the ray parameter p in favour of the off-shore coordinate y, then the solutions for the decay exponents are

$$\phi^{i} = -\frac{1}{4X}\left[\int_{y_0}^{y} \frac{\mathrm{d}y'}{F^{\frac{1}{2}}}\right]^2, \quad \phi^{t} = -\frac{1}{4X}\left[\int_{0}^{y} \frac{\mathrm{d}y'}{F^{\frac{1}{2}}} + \int_{0}^{y_0} \frac{\mathrm{d}y'}{F^{\frac{1}{2}}}\right]^2. \tag{6.11a, b}$$

If we retain only the cross-flow component D_{22} of the diffusivity tensor $\boldsymbol{D}$, then the equation (2.4*b*) for the amplitude factor z becomes

$$h\left(\frac{u}{m_1}\partial_x z - 2\frac{D_{22}}{m_2^2}\partial_y \phi \partial y z\right) - \frac{z}{m_1 m_2}\partial_y\left(h\frac{m_1}{m_2} D_{22}\partial_y \phi\right) = 0. \tag{6.12}$$

In terms of the X, F, δ notation this equation can be re-written

$$h\frac{u}{m_1}\partial_X z + \frac{z}{2m_1 m_2}\partial_y\left(hm_2 u\left(\frac{F}{F_0}\right)^{\frac{1}{2}}\right) = 0. \tag{6.13}$$

Mass conservation (2.1*b*) and the equation (6.7) for the ray separation J enable us to replace the y-derivative by X-derivatives:

$$\frac{1}{z}\partial_X z + \frac{1}{2hm_2 u}\partial_X(hm_2 u) + \frac{1}{2J}\partial_X J = 0. \tag{6.14}$$

Integration with respect to X yields the result

$$z^2 hm_2 uJ = \frac{\text{constant}}{4\pi} \quad \text{along rays.} \tag{6.15}$$

For the rays going outwards from the discharge, we can relate the ray constant to the value of the discharge rate Q and the flow properties at the discharge

$$\text{constant}^{i} = \frac{Q^2 m_2}{hD_{22} m_1} \quad \text{evaluated at} \quad x_0, y_0. \tag{6.16}$$

Conveniently, the p-labelling of the rays ensures that the constant is the same for all rays. Beyond the shoreline caustic the sign of J reverses, so the ray constant also needs to be reversed

$$\text{constant}^{\mathrm{t}} = -\frac{Q^2 m_2}{hD_{22} m_1} \quad \text{evaluated at} \quad x_{0,\,y_0}. \tag{6.17}$$

For the special case in which F is independent of x, the solutions (6.9d) and (6.10d) for the ray separations J^{i}, J^{t} imply that

$$z^{\mathrm{i}} = z^{\mathrm{t}} = \frac{(\text{constant}^{\mathrm{i}})^{\frac{1}{2}}}{2\pi^{\frac{1}{2}} X^{\frac{1}{2}} (hm_2\, u)^{\frac{1}{2}}} \left(\frac{F_0}{F}\right)^{\frac{1}{4}}. \tag{6.18}$$

Near the shoreline F, h and u tend to zero at the rates y, y and $y^{\frac{1}{2}}$. Thus, the ray amplitudes become singular at the rate y^{-1}. It is the purpose of the earlier sections of this paper to resolve the local failure of the ray method.

7. Illustrative example

To allow a direct comparison with Kay's (1987) exact solution we consider x-independent flow along a straight coastline with longitudinal (or skew) diffusion neglected. For turbulent flow over a smooth bed, the y-dependence of the depth topography, flow velocity and traverse diffusivity are related:

$$h = Hf(y), \quad u = Uf^{\frac{1}{2}}, \quad \epsilon D_{22} = Df^{\frac{3}{2}}. \tag{7.1a–c}$$

Here H, U and D are the values of the water depth, longshore velocity and transverse diffusivity at some convenient reference position (which might be the discharge distance $y = y_0$). The non-dimensional function $f(y)$ gives the shape of the depth profile outwards from the shoreline $y = 0$. Kay (1987, equation (25)) considers the special case $f = y$.

The validity of the ray solution (2.2) or the uniform solution (4.1a, b) depends upon a disparity of scales, as formalized by the parameter ϵ. If L is the off-shore lengthscale of the depth profiles, then the smallnes of the diffusion relative to the advection can be characterized:

$$\epsilon = \frac{D}{UL}. \tag{7.2}$$

If the coordinates x, y are scaled relative to the intrinsic lengthscale L, then the metric coefficients m_1, m_2 in the two directions have the values

$$m_1 = L, \quad m_2 = L. \tag{7.3}$$

For the ray solution detailed in the previous section the diffusivity/velocity (and metric coefficient) ratio F is given by

$$F = f(y). \tag{7.4}$$

The solutions (6.11a, b) for the decay exponents are

$$\phi^{\mathrm{t}} = -\frac{1}{4X}\left[\int_0^y \frac{\mathrm{d}y'}{f^{\frac{1}{2}}} - \int_0^{y_0} \frac{\mathrm{d}y'}{f^{\frac{1}{2}}}\right]^2, \quad \phi^{\mathrm{t}} = -\frac{1}{4X}\left[\int_0^y \frac{\mathrm{d}y'}{f^{\frac{1}{2}}} + \int_0^{y_0} \frac{\mathrm{d}y'}{f^{\frac{1}{2}}}\right]^2, \tag{7.5a, b}$$

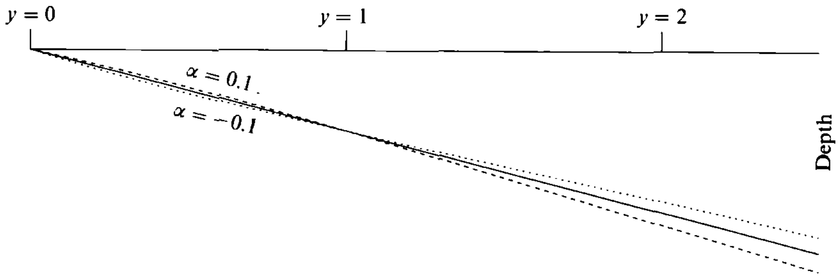


FIGURE 4. Three depth profiles for which the concentration contours are given in figures 5(*a*, *b*).

where X is the increment of x downstream of the discharge position x_0. For a volume discharge rate Q the solutions (6.18) for the amplitude factors are

$$z^{\mathrm{i}} = z^{\mathrm{t}} = \frac{Q}{2HUL(\pi X)^{\frac{1}{2}} f(y_0) f(y)}. \tag{7.6}$$

This ray solution has singularities as y_0^{-1} or y^{-1} if either the discharge site y_0 or observation position y is close to the shoreline.

The uniform solution (4.1*a*, *b*) is designed to resolve the spurious shoreline singularities. The ray combinations (5.4*a*–*d*) yield the canonical variables and amplitudes

$$s = -\frac{1}{4X}\left\{\left[\int_0^y \frac{\mathrm{d}y'}{f^{\frac{1}{2}}}\right]^2 + \left[\int_0^{y_0} \frac{\mathrm{d}y'}{f^{\frac{1}{2}}}\right]^2\right\}, \quad r = \frac{1}{4X^2}\left[\int_0^y \frac{\mathrm{d}y'}{f^{\frac{1}{2}}}\right]^2\left[\int_0^{y_0} \frac{\mathrm{d}y'}{f^{\frac{1}{2}}}\right]^2, \tag{7.7a, b}$$

$$A^{(0)} = \frac{Q}{8HUL\pi^{\frac{1}{2}}X^{\frac{5}{2}}}\left[\int_0^y \frac{\mathrm{d}y'}{f^{\frac{1}{2}}}\right]^2 \frac{1}{f(y)}\left[\int_0^{y_0} \frac{\mathrm{d}y'}{f^{\frac{1}{2}}}\right]^2 \frac{1}{f(y_0)}, \quad B^{(0)} = 0. \tag{7.7c}$$

Although the ray amplitude (6.6) was singular at the shoreline, the amplitude $A^{(0)}$ remains finite. The absence of the $B^{(0)}$ term means that at leading order in ϵ, the uniform solution

$$c = \epsilon^{-\frac{5}{2}} A^{(0)} \exp(\epsilon^{-1} s) K(\epsilon^{-2} r) \tag{7.8}$$

only differs from Kay's (1987, equation (35)) exact solution in the specification of the canonical variables r, s. (At this stage there is no need for the technical device of the $\pm$ sign.)

Figure 4 shows the three depth profiles $\alpha = 0,\ 0.1,\ -0.1$ in the family

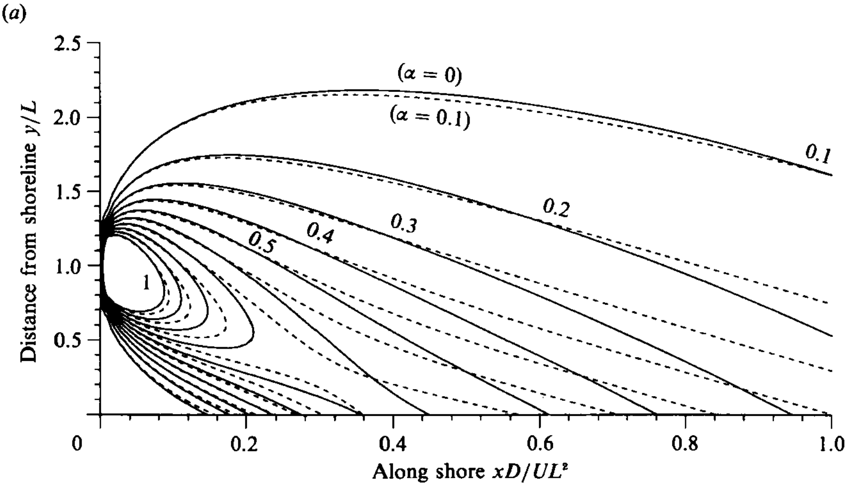
$$f(y;\alpha) = \frac{y}{[1+\alpha\phi(y)]^2}, \quad \text{with} \quad \psi(y) = (1-y)\exp(-\tfrac{1}{2}y). \tag{7.9a, b}$$

Kay's (1987) exact solution concerns the case $\alpha = 0$, when the beach has a uniform slope. The formulae (7.7*a*–*c*) yield the expressions

$$s = -\frac{1}{X}\{y[1+\alpha\Psi(y)]^2 + y_0[1+\alpha\Psi(y_0)]^2\}, \tag{7.10a}$$

$$r = 4\frac{yy_0}{X^2}[1+\alpha\Psi(y)]^2[1+\alpha\Psi(y_0)]^2, \tag{7.10b}$$

$$A^{(0)} = \frac{2Q}{HUL\pi^{\frac{1}{2}}X^{\frac{5}{2}}}[1+\alpha\psi(y)]^2[1+\alpha\Psi(y)]^2\ [1+\alpha\psi(y_0)]^2\ [1+\alpha\Psi(y_0)]^2 \tag{7.10c}$$



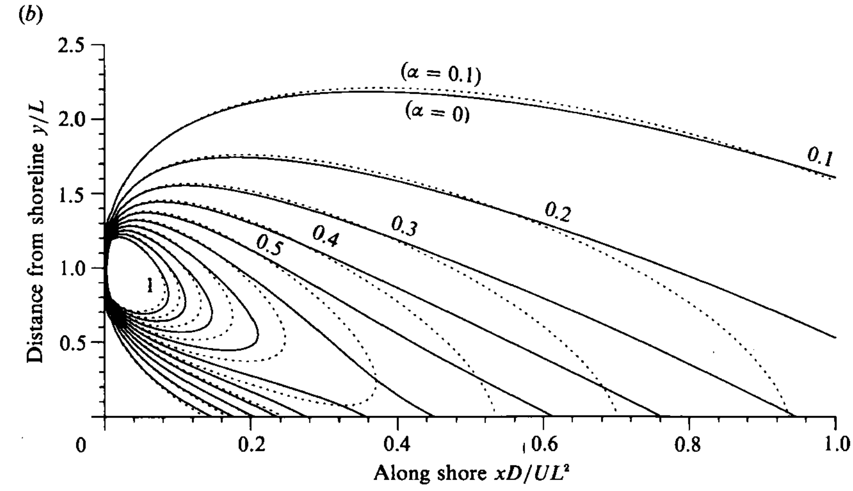


FIGURE 5. (*a*) Comparison between the concentration contours —— when the beach has uniform slope, and the contours ----- when there is slightly reduced depth shoreward of the discharge. (*b*) Comparison between the concentration contours —— when the beach has uniform slope, and the contours ..... when there is slightly increased depth shoreward of the discharge.

with
$$\Psi(y) = \frac{1}{2y^{\frac{1}{2}}}\int_0^y \frac{\psi(Y)}{Y^{\frac{1}{2}}}\mathrm{d}Y = \exp\left(-\tfrac{1}{2}y\right). \tag{7.10d}$$

For more complicated depth profiles the integrals (7.7*a–c*) would need to be evaluated numerically.

Figure 5(*a*) compares Kay's (1987) exact solution for $\alpha = 0$, with the uniform asymptotic approximation for $\alpha = 0.1$ when the discharge has strength

$$Q = \frac{HUL}{\epsilon} = \frac{HU^2L^2}{D}, \tag{7.11}$$

and is situated at the reference distance $y_0 = 1$ from the shoreline. At this location the initial depth, velocity and diffusivity is the same for all values of α. Figure 5(*b*) gives the corresponding comparison between the cases $\alpha = 0$, $\alpha = -0.1$. As the

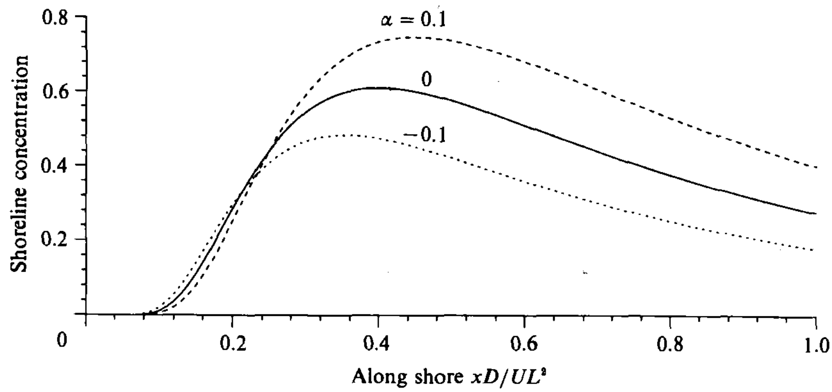


FIGURE 6. Shoreline concentrations for the three depth profiles shown in figure 4.

plumes widen and experience differences in depths, velocities and diffusivities, the solutions for the different topographies begin to separate. Shallower water is associated with slower mixing and higher concentrations. So, in figure 5(*a*) the higher (and later) shoreline concentrations occur for the shallower beach $\alpha = 0.1$. Similarly, in figure 5(*b*) it is the uniform beach $\alpha = 0$ that is associated with the higher (and later) shoreline concentrations. Figure 6 shows the shoreline concentrations for the three values of α. The differences in shoreline concentration contours are much more marked than are the differences in depth profiles.

In practice the reference value D of the transverse diffusivity scales as

$$D \approx 0.2HU_*, \tag{7.12}$$

where H is the depth and U_* is the friction velocity at the reference position (Fischer *et al.* 1979, equation (5.4)). Thus, using (7.2) we can estimate the small parameter ϵ

$$\epsilon \approx 0.2\left(\frac{U_*}{U}\right)\frac{H}{L}. \tag{7.13}$$

So, if we estimate the velocity ratio U_*/U as being about 0.1, and the beach slope H/L as being about 0.1, then we arrive at the estimate

$$\epsilon \approx 0.002. \tag{7.14}$$

We recall that the asymptotic expansions (4.4*a–d*) proceed in powers of ϵ^2, so the approximate formula (7.8) will be extremely accurate.

8. Summary

The ray approximation corresponds to regarding any contaminant plume in a shallow-water flow as being a distortion of the Gaussian plume in a straight constant depth and velocity flow. Provided that the plume is long and narrow, it is an easy computational task to evaluate the necessary distortions (see §6). Unfortunately, there are spurious singularities at a beach where the depth and velocity tend to zero. The new non-singular beach solution is instead a distortion of Kay's (1987) exact solution for a discharge in the idealized case of flow along a straight uniformly-sloping beach.

For the same contaminant plume the two alternative distortions are necessarily related (see equations (5.4*a–d*)). Conveniently, the ray information can be used to construct the dependent variables r, s and amplitudes $A^{(0)}$, $B^{(0)}$ in the non-singular beach solution. For simple beach geometries the singular ray solutions, and hence the non-singular beach solution can be obtained analytically.

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